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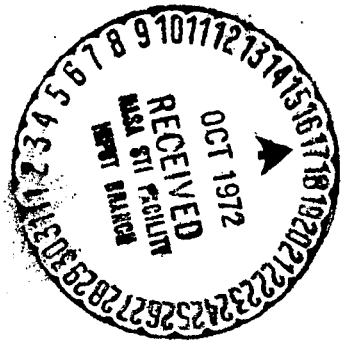

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QUANTUM THEORY OF THE GLOWING ELECTRON. I

The approximate quantum theory of the radiation of fast electrons, moving in a magnetic field, is considered here. The conditions of the appearance of quantum corrections of the order of n^2/l where n is the number of the radiated harmonic, and l is the azimuthal quantum number, are investigated. It is shown that these corrections contract when the total radiation intensity is calculated. Quantum corrections of the order of n^2/l remain in the total intensity.

1. As a known, the classical theory of the glowing electron /1-4/, developed recently, describes well the observed glow during the motion of relativistic electrons in a magnetic field /5/. The classical theory of the influence of the earth's magnetic field on the large Auer showers was given in /6/.

Developing the quantum theory of the glowing electron /7-4/, we can find quantum corrections for the radiation frequency, the motion trajectory, and the radiation intensity. In /7/ it was shown that quantum corrections for the radiations frequency have an order of n/l, which corresponds to the energy

$$E \sim mc^2 (amc/\hbar)^{1/3}, \qquad (1)$$

where a is the radius of the orbit, m is the electron mass, l is the azimuthal quantum number (with $\hbar l = Ea\beta/2c$), and n is the number of the radiated harmonic. In the same work it was shown that purely classical transitions between circular orbits (the initial radial quantum number s and the final s'

are equal to zero) are determined by a condition stronger than (1): $n^2/l \ll 1$, or

$$E \sim mc^2 (amc/\hbar)^{1/4}. \qquad (2)$$

When the value n^2/l becomes comparable to unity, the appearance of quantum transitions from a circular ($s = 0$) orbit to a noncircular one ($s' \neq 0$) is possible. In work /7/ the probabilities of such transitions were calculated with a precision of up to a value of n^2/l. There it was also indicated that quantum corrections for the radiation intensity have an order of n^2/l. Nevertheless, from the final formulas cited in the same work it follows that, when the total (i.e. summated for the final value of the radial quantum numbers s') radiation intensity is calculated, terms of the order of n^2/l contract. Therefore quantum corrections of the order of n^2/l can be observed only if it were possible to distinguish phenomena connected with transitions between ciruclar orbits, from phenomena connected with transitions from a circular orbit to a noncircular one. All these refinements were made in /4/.

In the present work we want to determine the total radiation intensity, and then to examine in greater detail the problems connected with transitions between circular and noncircular orbits. When determining the total radiating intensity, it is simplest of all to calculate the problem in terms of Cartesian coordinates (see #2). But when determining the probability of individual transitions between orbits (#3), it is better to use cylindrical coordinates.

2. We want, first of all, to determine the total radiation intensity of the frequency $\omega_{n'n}$. For this purpose, limiting ourselves for the time being to a constant homogeneous magnetic field, we shall solve our problem by means of the Dirac equation in terms of Cartesian coordinates.

Let us take the potantial in the form of $A_y = Hx, \; A_x = A_z = 0.$ Then the wave functions of the electron is the solution of the Dirac equation

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = c\rho_1\vec{\sigma}\left(\frac{\hbar}{i}\vec{\nabla} + \frac{e}{c}\mathbf{A}\right)\psi + \rho_3 mc^2\psi \qquad (3)$$

and at the same time is an eigenfunction of the operator of the projection of the kinetic impulse on the spin $\vec{\sigma}\mathbf{P} = \vec{\sigma}\left(\frac{\hbar}{i}\vec{\nabla} + \frac{e}{c}\mathbf{A}\right)$, of the operator $\left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right)$ of the projection of the impulse on axis z, and of the operator of the x coordinate of the center of rotation $x_0 = \frac{1}{l}\frac{c\hbar}{eH}\frac{\partial}{\partial y}$, which commutate with the Hamiltonian. It is easy to see that the wave function ψ has the form (see also /8-9/:

$$\psi = \frac{1}{\sqrt{L_y L_z}} e^{-i\varepsilon\kappa t + ik_2 y + ik_3 z - (\xi^2/2)} \begin{Bmatrix} -s\alpha(\varepsilon)\, A(s)\, H_{l-1}(\xi) \\ \alpha(\varepsilon)\, B(s)\, H_l(\xi) \\ \varepsilon\beta(\varepsilon)\, A(s)\, H_{l-1}(\xi) \\ -s\varepsilon\beta(\varepsilon)\, B(s)\, H_l(\xi) \end{Bmatrix}, \qquad (4)$$

where

$$\alpha(\varepsilon) = \sqrt{\frac{1}{2}\left(1 + \varepsilon\frac{k_0}{K}\right)}, \quad \beta(\varepsilon) = \sqrt{\frac{1}{2}\left(1 - \varepsilon\frac{k_0}{K}\right)}, \qquad (5)$$

$$A(s) = \sqrt{\frac{1}{2}\left(1 + s\frac{k_3}{k}\right)}, \quad B(s) = \sqrt{\frac{1}{2}\left(1 - s\frac{k_3}{k}\right)}. \qquad (6)$$

$H_l(\xi)$ is a polynomial, proportional to the Chevyshev-Hermite polynomial

$$H_l(\xi) = \sqrt{\frac{1}{2^l l!}}\sqrt{\frac{\gamma}{\pi}}\,H_l(\xi), \qquad (7)$$

and furthermore, $\xi = x\sqrt{\gamma} + k_2/\sqrt{\gamma}$, $\gamma = eH/ch$. The energy of the electron is equal to $E = \varepsilon ch K = \varepsilon ch\sqrt{k_0^2 + k_3^2 + 2l\gamma}$, $\varepsilon = \pm 1$, l is an interger, $l = 0, 1, \ldots$, the projection of the impulse on the spin equals shk, $s = \pm 1$, $k = \sqrt{k_3^2 + 2l\gamma}$, the impulse along the field equals $\hbar k_3$ and the x coordinate of the center of roration equals k_2/γ. In comparison with the usual representation of the matrices we have replaced σ_x with σ_y and σ_y with σ_x. Thus we have achieved the elimination of imaginary values from matrices which characterize spin states.

Let us now take as a perturbation the energy of the interaction of the electron with a radiated electromagnetic wave

$$U = \frac{-e}{\sqrt{L_x L_y L_z}}\sqrt{\frac{2\pi c\hbar}{\varkappa}}\,\vec{\alpha}\,a_\lambda^+\,e^{ic\varkappa t - i\vec{\varkappa}\vec{r}} \qquad (8)$$

The properties of the Fourier amplitude of the photon field are presented, for example, in /10/. The radiation frequency and the energy of the photon are respectively equal to $c\varkappa$ and $c\hbar\varkappa$. Using the conventional apparatus of perturbation theory, we find for the radiation intensity, averaging with respect to the initial spin states of the electron and summating for the polarization states of the photon:

$$W = \frac{e^2}{4\pi}\sum_{l'=0}^{l}\int d\Omega_{\varkappa}\sum_{s,\,s'=\pm 1}\left\{|\vec{\varkappa}|^2 - (\vec{\varkappa}\vec{\varkappa}^{(0)})^2\right\}\frac{K-\varkappa}{K-\varkappa\sin^2\theta}. \qquad (9)$$

By virtue of the laws of the conservation of energy $K = K' + \varkappa$ and of the impulse along the field $k_3 = k_3' + \varkappa \cos\theta$, the value $\varkappa_1$ with the condition $k_3 = 0$, is equal to

$$\varkappa = \frac{\sqrt{2l\gamma}}{\beta \sin^2\theta}\left(1 - \sqrt{1 - \frac{n\beta^2 \sin^2\theta}{l}}\right), \qquad (10)$$

where $\beta = \sqrt{2l\gamma}/K$; besides, we have the condition $k_2 = k_2' + \varkappa \sin\theta \sin\varphi$ (change of the x-coordinate of the center of rotation during radiation of the photon). The projections of the matrix elements of the value $\bar{\alpha} e^{i\varkappa_1 x}$ will be equal to

$$\bar{\alpha}_x = i\{-ss'e\alpha'\beta A'BI(l'-1, l) + e\alpha'\beta B'AI(l', l-1) - e'\beta'\alpha A'BI(l'-1, l) + ss'e'\beta'\alpha B'AI(l', l-1)\}, \qquad (11)$$

$$\bar{\alpha}_y = -ss'e\alpha'\beta A'BI(l'-1, l) - e\alpha'\beta B'AI(l', l-1) - e'\beta'\alpha A'BI(l'-1, l) - ss'e'\beta'\alpha B'AI(l', l-1), \qquad (12)$$

$$\bar{\alpha}_z = -s'e\alpha'\beta A'AI(l'-1, l-1) + se\alpha'\beta B'BI(l', l) - se'\beta'\alpha A'AI(l'-1, l-1) + s'e'\beta'\alpha B'BI(l', l), \qquad (13)$$

with $s = sd = 1$, and

$$I(l', l) = \frac{\sqrt{\gamma}}{\sqrt{\pi l!\, l'!\, 2^{l+l'}}} \int_{-\infty}^{\infty} H_{l'}\left(x\sqrt{\gamma} - \frac{\varkappa_2}{\sqrt{\gamma}}\right) H_l(x\sqrt{\gamma}) \times \exp\left\{-\frac{1}{2}\left(x\sqrt{\gamma} - \frac{\varkappa_2}{\sqrt{\gamma}}\right)^2 - \frac{\gamma}{2}x^2 + i\varkappa_1 x\right\} dx =$$

$$= \frac{1}{\sqrt{l!\,l'!}} \exp\left\{-\frac{x_1^2+x_2^2}{4\gamma} + i\frac{x_1x_2}{2\gamma} + i(l-l')\left(\frac{\pi}{2} - \operatorname{arc\,tg}\frac{x_2}{x_1}\right)\right\} \times$$
$$\times \left(\frac{x_1^2+x_2^2}{2}\right)^{(l-l')/2} Q_{l'}^{(l-l')}\left(\frac{x_1^2+x_2^2}{2\gamma}\right), \qquad (14)$$

where $Q_{l'}^{(l-l')}$ is a generalized Sonin-Laguerre polynomial.

It is easy to verify that (9) does not depend on the angle $\varphi = \operatorname{arc\,tg} x_2/x_1$. Therefore we immediately assume $\varphi = \pi/2$, as a result of which we shall have for planar motion $(n/l \sim 0)$

$$W = \frac{ce^2}{4\pi} \sum_{l'=0}^{l} \int d\Omega \sum_{s,\,s'=\pm 1} \{\cos^2\theta\,|\bar{\alpha}_y|^2 + |\bar{\alpha}_x|^2\}\,\varkappa^2 \frac{K-\varkappa}{K-\varkappa\sin^2\theta}. \qquad (15)$$

Summating for the spin states of the electron (when $k_y = 0$), we shall have

$$\sum_{s,\,s'=\pm 1} |\bar{\alpha}_x|^2 = \frac{KK'-k_0^2}{2KK'}\left(|I(l',l-1)|^2 + |I(l'-1,l)|^2\right) -$$
$$- \frac{\gamma\sqrt{ll'}}{KK'}\left(I^+(l',l-1)\,I(l'-1,l) + I^+(l'-1,l)\,I(l',l-1)\right), \qquad (16)$$

$$\sum_{s,\,s'=\pm 1} |\bar{\alpha}_y|^2 = \frac{KK'-k_0^2}{2KK'}\left(|I(l',l-1)|^2 + |I(l'-1,l)|^2\right) +$$
$$+ \frac{\gamma\sqrt{ll'}}{KK'}\left(I^+(l',l-1)\,I(l'-1,l) + I^+(l'-1,l)\,I(l',l-1)\right), \qquad (17)$$

$$\sum_{s,\,s'=\pm 1} |\bar{\alpha}_z|^2 = \frac{KK' - k_0^2}{2KK'} \left(|I(l', l)|^2 + |I(l'-1, l-1)|^2\right) - \frac{\gamma\sqrt{ll'}}{KK'} \left(I^+(l', l)\, I(l'-1, l-1) + I^+(l'-1, l-1)\, I(l', l)\right) \sim \left(\frac{n}{l}\right)^2 = 0, \quad (18)$$

whence

$$W = \frac{ce^2}{2} \sum_{l'=0}^{l} \int_0^{\pi} \sin\vartheta d\vartheta \varkappa^2 \left\{ \frac{KK' - k_0^2}{2KK'} \left(|I(l', l-1)|^2 + |I(l'-1, l)|^2\right) - \frac{\gamma\sqrt{ll'}}{KK'} \left(I^+(l', l-1)\, I(l'-1, l) + I^+(l'-1, l)\, I(l', l-1)\right) + \cos^2\vartheta \left[\frac{KK' - k_0^2}{2KK'} \left(|I(l', l-1)|^2 + |I(l'-1, l)|^2\right) + \frac{\gamma\sqrt{ll'}}{KK'} \left(I^+(l', l-1)\, I(l'-1, l) + I^+(l-1, l)\, I(l', l-1)\right) \right] \right\} \frac{K - \varkappa}{K - \varkappa\sin^2\vartheta}. \quad (19)$$

We now take into account the fact that in this case $\frac{1}{l} = \frac{2}{\beta}\frac{mc^2}{E}\frac{\hbar}{amc} \ll 1$, and, in addition, assume that $n^2 \equiv (l - l')^2 \ll l$. Then

$$\varkappa = n\beta\sqrt{\frac{\gamma}{2l}}\left(1 + \frac{n\beta^2\sin^2\vartheta}{4l}\right), \quad (20)$$

$$\frac{\varkappa^2\sin^2\vartheta}{2\gamma} = \frac{n^2\beta^2\sin^2\vartheta}{4l}\left(1 + \frac{n\beta^2\sin^2\vartheta}{2l}\right), \quad (21)$$

$$\frac{KK' - k_0^2}{2KK'} \approx \frac{\gamma\sqrt{ll'}}{KK'} \approx \frac{\beta^2}{2}\left(1 - \frac{n}{2l}(1 - \beta^2)\right). \quad (22)$$

Here,

$$W_n(0)=\frac{ce^2\beta^4n^3}{4a^3}\left[1+\frac{n}{2l}\cdot(2\beta^2\sin^2\theta-1)\right]\{\cos^2\theta(|I(l',l-1)|+|I(l'-1,l)|)^2+$$
$$+(|I(l',l-1)|-|I(l'-1,l)|)^2\}. \qquad (23)$$

We now derive an asymptotic formula for $Q_{l-m}^{(p)}(z)$ which is valid under the condition that m and p be small in comparison to c 1 and 1/z. We make the substitution $z=\zeta/l$ and make use of integral representations for Sonin-Laguerre polynomials and Bessel functions:

$$Q_{l-m}^{(p)}\left(\frac{\zeta}{l}\right)=\frac{(l-m+p)!}{2\pi i}\int_C e^t\left(1-\frac{\zeta}{lt}\right)^{l-m}\frac{dt}{t^{p+1}},$$
$$\zeta^{-p/2}J_p(2\sqrt{\zeta})=\frac{1}{2\pi i}\int_C e^{t-\zeta/t}\frac{dt}{t^{p+1}}, \qquad (24)$$

where the contour C goes from $-\infty$, bypasses t = 0 and $t=\zeta/l$ in the positive direction, and goes away into $-\infty$. . If we take into account that

$$\left(1-\frac{\zeta}{lt}\right)^l\approx e^{-\zeta/t}\left(1-\frac{\zeta^2}{2lt^2}\right), \qquad (25)$$
$$\left(1-\frac{\zeta}{lt}\right)^{-m}\approx 1+\frac{m\zeta}{lt}, \qquad (26)$$

we shall obtain, with a precision of up to terms of the order of n^2/l

$$Q_{l-m}^{(p)}\left(\frac{\zeta}{l}\right)=(l-m+p)!\times$$
$$\times\left\{\zeta^{-p/2}J_p(2\sqrt{\zeta})+\frac{m}{l}\zeta^{-(p/2)+1/2}J_{p+1}(2\sqrt{\zeta})-\frac{1}{2l}\zeta^{-(p/2)+1}J_{p+2}(2\sqrt{\zeta})\right\}. \qquad (27)$$

We further take into account that, with the same precision,

$$J_p\left[n\beta\sin\theta\left(1+\frac{n\beta^2\sin^2\theta}{4l}\right)\right]=J_p(n\beta\sin\theta)+\frac{n^2\beta^2\sin^2\theta}{4l}J'_p(n\beta\sin\theta) \qquad (28)$$

and

$$\sqrt{\frac{l'}{(l-n)!}l^{-n}}=\sqrt{\frac{(l-1)!}{(l-n-1)!}l^{-n}}=1-\frac{n^2}{4l}, \qquad (29)$$

we shall find

$$(p=m=n)\quad |I(l',l)|=J_n(n\beta\sin\theta)-\frac{n^2\beta\sin\theta}{4l}(1-\beta^2\sin^2\theta)J'_n(n\beta\sin\theta), \qquad (30)$$

$$(p=n,\ m=n+1)\quad |I(l'-1,l-1)|=J_n(n\beta\sin\theta)-$$
$$-\frac{n^2\beta\sin\theta}{4l}(1-\beta^2\sin^2\theta)J'_n(n\beta\sin\theta), \qquad (31)$$

$$(p=n-1,\ m=n)\quad |I(l',l-1)|=J_{n-1}(n\beta\sin\theta)+$$
$$+\frac{n^2}{4l}[\beta\sin\theta J_n(n\beta\sin\theta)-J_{n-1}(n\beta\sin\theta)+\beta^3\sin^3\theta J'_{n-1}(n\beta\sin\theta)], \qquad (32)$$

$$(p=m=n+1)\quad |I(l'-1,l)|=J_{n+1}(n\beta\sin\theta)+$$
$$+\frac{n^2}{4l}[\beta\sin\theta J_{n+2}(n\beta\sin\theta)-J_{n+1}(n\beta\sin\theta)+\beta^3\sin^3\theta J'_{n+1}(n\beta\sin\theta)], \qquad (33)$$

having made use of recurrent relationships between Bessel functions. Collecting all the terms, we shall obtain with the same precision

$$W_n(0) = \frac{ce^2\beta^2 n^2}{a^2}\left\{\operatorname{ctg}^2\theta J_n^2(n\beta\sin\theta) + \beta^2 J_n'^2(n\beta\sin\theta) - \right.$$
$$\left. - \frac{n^2}{2l}\beta\frac{2-(1+\beta^2)\sin^2\theta}{\sin\theta}(1-\beta^2\sin^2\theta)\,J_n(n\beta\sin\theta)\,J_n'(n\beta\sin\theta)\right\}. \quad (34)$$

Taking into account the fact that, for the most intensively radiated harmonic,

$$\cos^2\theta \sim 1-\beta^2 \sim n^{-2/3} \ll 1 \quad \text{и} \quad J_n(n) \sim n^{-1/3}, \quad J_n'(n) \sim n^{-2/3}, \quad (35)$$

it is easy to find that the ratio of the corrective term to the basic one has an order of n/l.

Analogously, it is easy to show that terms containing n^3/l^2, n^4/l^3, etc., do not yield corrections higher than n^2/l^3. Thus we arrive again at the condition (1) - 1) Note in proofreading. At present the authors have succeeded in calculating the value of the total radiation intensity with account taken of the terms of the order of n/l. $W = \frac{2}{3}\frac{e^2c}{a^2}\left(\frac{E}{mc^2}\right)^4\left[1-\frac{55\sqrt{3}}{16}\frac{\hbar}{mca}\left(\frac{E}{mc^2}\right)^2+\ldots\right]$.

3. Let us now turn to the calculation of quantum corrections with account taken of heterogeneity of the magnetic field. We shall examine the motion of the electron in an axially-symmetric magnetic field, which is constant in time, and which depends on the radius in the vicinity of a stationary orbit of the form $H = \text{const}\, r^{-q}, 0<q<1$. Disregarding spin effects, we limit ourselves in solving the problem to the relativistic wave equation of Klein - Fok. We direct the magnetic field along the

z axis:

$$A_y = \frac{xH_0}{r^q}, \quad A_x = -\frac{yH_0}{r^q}, \quad H_z = \frac{H_0(2-q)}{r^q}, \quad H_0 = \text{const} \qquad (36)$$

(in the vicinity of a stationary orbit). Here the Klein - Fok equation has the form:

$$(E^2 - c^2P^2 - c^2A^2 - 2ecA\mathrm{P} - m^2c^4)\psi = 0. \qquad (37)$$

Restricting ourselves to the study of planar movement, we introduce, for convenience, polar coordinates. It is natural to look for the solution of equation (37) in the form:

$$\psi = \frac{e^{il\varphi}}{\sqrt{2\pi}} R(r), \qquad (38)$$

taking into account that ψ is a cyclic coordinate $(l = 0, 1, 2, \ldots)$. Then, for the radial functions R, we obtain the equation

$$R'' + \frac{1}{r}R' + \left(\frac{E^2 - m^2c^4}{c^2\hbar^2} - \frac{e^2H_0^2}{c^2\hbar^2}r^{2(1-q)} - \frac{2eH_0l}{c\hbar}r^{-q} - \frac{l^2}{r^2}\right)R = 0, \qquad (39)$$

which, when the function $\sqrt{mc^2/E}\, u = \sqrt{r}R$ is introduced, is reduced to the form:

$$u'' + f(r)u = 0. \qquad (40)$$

Let us further expand the function f(r) into a Taylor series according to degrees of deviation from the position of equilibrium r - a, where a is defined

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from the condition of extermum f(r):

$$a = \left(\frac{i}{\gamma(1-q)}\right)^{1/(2-q)}, \quad \gamma = \frac{eII_0}{ch}, \quad \alpha = f(a), \quad \beta^2 = -\frac{1}{2}f''(a). \qquad (41)$$

Assuming that the deviation $x = r - a$ will be small $(x/a \ll 1)$, we shall limit ourselves to a term which is quadratic with respect to x; then for radial functions u we shall obtain the equation of a harmonic oscillator:

$$u'' + (\alpha - \beta^2 x^2)\, u = 0. \qquad (42)$$

The solution for this problem are the orthonormal Chebyshev-Hermite functions

$$u_s = \sqrt[4]{\frac{\beta}{\pi}} \frac{1}{\sqrt{2^s s!}} e^{-(\beta^2/2)(r-a)^2} H_s(\sqrt{\beta}(r-a)), \qquad (43)$$

where H_3 is a Chevyshev-Hermite polynomial, and, furthermore α/β takes the discrete values

$$\alpha/\beta = 2s + 1, \quad s = 0, 1, 2, \ldots \qquad (44)$$

From the latter relationship it is easy to find the eigen values for energy:

$$\frac{E^2 - m^2c^4}{c^2h^2} = (2-q)\,\gamma^{\frac{2}{2-q}}\left[l\,\frac{2-q}{1-q} + (2s+1)\sqrt{1-q}\right], \qquad (45)$$

and for the radiation frequency we find the expression:

$$\omega_n = \bar{n}\omega_0 (1 + O(n/l)), \quad \omega_0 = v/a, \quad \bar{n} = n + n'\sqrt{1-q}, \qquad n = l - l', \quad n' = s - s'. \tag{46}$$

In the classical approximation $(l \to \infty)$

$$\omega_n = nv/a.$$

We now make a remark concerning the shape of the orbit. It follows from equation (43) that small radial oscillations the amplitude of which can be determined from the state of positiveness of coefficient $\alpha - \beta^2 x^2$ and has the form:

$$x = \Delta a = \pm a \sqrt{\frac{2\sqrt{1-q}}{2-q}} \sqrt{\frac{s}{l}}. \tag{47}$$

are superimposed on the rorating motion of the electron with a circumference of the radius $a = (l/\gamma(1-q))^{1/(2-q)}$

In the classical approximation quantum number $l = (eH_0/ch)(1-q)a^{2-q}$ will be considerably larger than s $(h \to 0,\ l \to \infty)$. Therefore the "washiness" of the radius disappears, and the motion becomes circular.

We shall consider the problem of the electron radiation as its interaction with a field of virtual photons; here we make calculations by the method of the perturbation theory, limiting ourselves to the first approximation and retaining in the perturbation energy only linear terms with respect to the vector-potential $\mathbf{A}^{\Phi}$, which characterizes a photon field:

$$\mathbf{A}^{\Phi} = \frac{1}{L^{3/2}} \sum_{\vec{\varkappa}} \sqrt{\frac{2\pi ch}{\varkappa}} \left(a^{+} e^{i c\varkappa t - i\vec{\varkappa}\vec{r}} + a e^{-i c\varkappa t + i\vec{\varkappa}\vec{r}}\right). \tag{48}$$

Then, for the differential value of radiation intensity in a unit of time inside the solid angle $d\Omega$ we obtain the expression

$$dW_n = \frac{\omega_n^2 e^2 c}{2\pi E^2} (\mathbf{a}\bar{\mathbf{P}})(\mathbf{a}^+\bar{\mathbf{P}})\, d\Omega, \qquad (49)$$

where $\bar{\mathbf{P}}$ is a matrix element of the impulse operator

$$\bar{\mathbf{P}} = \int \psi'^+ e^{-i\vec{\varkappa}\mathbf{r}} \mathbf{P}\psi\, d\mathbf{r}, \qquad \mathbf{P} = \frac{\hbar}{i}\vec{\nabla} + \frac{e}{c}\mathbf{A},$$

and the wave functions ψ'^+ and ψ are respectively equal to

$$\psi'^+ = \frac{e^{-il'\varphi}}{\sqrt{2\pi}} R(l', s'), \quad \psi = \frac{e^{il\varphi}}{\sqrt{2\pi}} R(l, s). \qquad (50)$$

By chosing spherical coordinates for the $\vec{\varkappa}^0$ the unitary vector which characterizes the radiation direction, and cylindrical coordinates for P, and directing $\vec{\varkappa}^0$ into the plane yz, we find

$$(\mathbf{a}\bar{\mathbf{P}})(\mathbf{a}^+\bar{\mathbf{P}}) = \bar{P}_x^2 + \cos^2\theta \bar{P}_y^2. \qquad (51)$$

Here we disregard the projection of vector $\vec{\varkappa}^0$ on to the z axis in comparison with the other projections, on the basis of conclusions of the classical theory which state that the radiation originates basically in a plane of electron rotation $(\theta \approx \pi/2)$ (see /4/). Then finally,

$$dW_n = \frac{\omega_n^2 e^2}{2\pi E^2} (\bar{P}_x^2 + \cos^2\theta \bar{P}_y^2)\, d\Omega. \qquad (52)$$

Turning to polar coordinates and disregarding the influence of the impulse operator in a radial direction (r is replaced by the value of a, in which $|\psi|^2$ reaches the maximum), we obtain a simple expression for the matrix element of operators P_x and P_y:

$$P_x = -\frac{e}{c} H(a)\, a J'_{l-l'}\left(\frac{\omega_n}{c} a \sin\theta\right) A_{ss'},$$
$$P_y = \frac{e}{c} \frac{H(a)(l-l')c}{\omega_n \sin\theta} J_{l-l'}\left(\frac{\omega_n}{c} a \sin\theta\right) A_{ss'}, \qquad (53)$$

where $J_{l-l'}$ is a Bessel function,

$$A_{ss'} = \int_0^\infty rRR'dr \Big/ \sqrt{\int_0^\infty rR^2dr \int_0^\infty rR'^2 dr}, \qquad (54)$$

and the radiation intensity of n the harmonic (n = l - l') will be equal to

$$dW_n = \frac{ce^4 n^2 H^2(a) A_{ss'}^2}{2\pi E^2} [\beta'^2 J_n'^2(n\beta' \sin\theta) + J_n^2(n\beta' \sin\theta) \operatorname{ctg}^2\theta]\, d\Omega, \qquad (55)$$

where $\beta' = \frac{\omega_n}{n\omega_0}\frac{v}{c}$, $\beta = \frac{v}{c}$.

In order to find the spectral intensity distribution, we integrate (55) with respect to the solid angle $d\Omega$; after several transformations we obtain

$$W_n = \frac{cc^4 H^2(a)\, n}{\beta E^2}\left(1 - \frac{n'}{n}\sqrt{1-q}\right)\left[2\beta^2\left(1 + 2\frac{n'}{n}\sqrt{1-q}\right)J'_{2n}(2\beta\bar{n}) - \right.$$
$$\left. - \left(1 - \beta^2 - 2\frac{n'}{n}\sqrt{1-q}\,\beta^2\right)\int_0^{2\beta\bar{n}} J_{2n}(x)\,dx\right] A^2_{ss'}. \qquad (56)$$

Here we have used the fact that $\omega_n = \bar{n}\omega_0 = (n + n'\sqrt{1-q})\,\omega_0$.

In the ultra-relativistic case n'/n can be omitted in comparison to unity, then W_n differs from the respective classical expression only by the coefficient $A^2_{ss'}$:

$$W_n = W_n^{KA} A^2_{ss'}, \qquad (57)$$

where the summation must be made for index s'.

The calculation of coefficient $A_{ss'}$, by means of the substitution of (43) in formula (54) yields

$$A_{ss'} = (-1)^{n'} \frac{z^{n'/2} e^{-z/2}}{\sqrt{s!\,s'!}} Q_s^{(n')}(z), \qquad (58)$$

where $Q_s^{(n')}$ are generalized Sonin-Laguerre polynomials, and $z = \frac{1}{(2-q)\sqrt{1-q}}\frac{\bar{n}^2}{2l}$.

If, for simplicity's sake, we examine radial transitions from the initial circular orbit $(s = 0,\ s' \neq 0)$, coefficient $A^2_{ss'}$ obtains the very simple expression:

$$A^2_{0s'} = e^{-z}\frac{z^{s'}}{s'!}. \qquad (59)$$

Summating $A^2_{ss'}$, for s', we obtain unity, thus terms of the order of n^2/l contract and do not influence the total intensity.

For the calculation of arbitrary transtions, replacing s by s', coefficient $A^2_{ss'}$, can be approximated precisely enough by the following

expression

$$A^2_{ss'} = J^2_{s-s'}(x), \qquad (60)$$

where $J_{s-s'}$ is a Bessel function and $x = \frac{n}{\sqrt{l}}\sqrt{\frac{s+s'+1}{(2-q)\sqrt{1-q}}}$.

Expanding (60) with respect to the value $n/\sqrt{l}$ and giving index s' the values $s' = s,\ s \pm 1,\ s \pm 2$, we come to the conviction that value $\sum_{s'} A^2_{ss'}$ is equal to unity.

4. The cited calculations show that the relationship of quantum corrections to classical terms in the total radiation intensity of the glowing electron has the order of n/l or $n\Lambda/\pi a$, where $\Lambda = 2\pi\hbar c/E$ is the de Broglie wave length of a moving electron. Corrections to the radiation frequency with a given value of n also have the order of n/l. Therefore the classical theory for determination of total radiation intensity can be applied at energies of

$$E \ll mc^2 (amc/\hbar)^{1/3}.$$

In a recently published article /11/, the author found quantum corrections for total radiation intensity of one electron. Here, his factor of the order of n^2/l (see /11/ formula (26a) appears even in the exponent. In connection with this, he came to the conclusion that quantum corrections already at comparatively low energies decrease the radiation intensity by several times. This conclusion is linked to a series of inaccuracies permitted by him in the approximate calculations. He assumes, for example, that

$$(l+n)!/l! = l^n,$$

when actually this value is equal to

$$(l+n)!/l! = l^n e^{n^2/2l}.$$

Besides, the author disregards in this work the transitions with a change of the radial quantum number s, considering that they will yield terms of the order of 1/l; in reality terms of the order of n^2/l appear during transitions from the circular orbit to a noncircular one, which was noted already in work /7/. In the summation of intensities for all radial transitions they contract, as it was shown in /4/.

Mr. A.I. Morozov took part in the discussion on a series of problems connected with the accounting of quantum corrections.

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